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共线方程线性化的矩阵模型

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摘要 借鉴计算机视觉投影方程的矩阵表达形式，将解析形式下的共线方程构造为矩阵方程表达，再以投影矩阵元素作为复合函数并基于矩阵分析方法，实现共线方程对各变量的统一求导。首先，与传统解析法线性化相比，矩阵分析过程工整，形式简洁，易于理解，便于应用线性库进行数值解算；其次，对于不同构造形式下的旋转矩阵，该方法都具有较好的适应性；最后，构建的共线方程的矩阵形式对于摄影测量借鉴计算机视觉方法也有重要启示意义。

关键词 共线方程; 投影方程; 投影矩阵; 齐次坐标

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Collinear Equation Linearized Matrix Model

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Abstract Using the matrix expression form of computer vision projection equation, the collinear equation is constructed into matrix equation. With the projection matrix element as a composite function, this paper realizes the unification derivation of each variable of the collinear equation based on the matrix analysis method. Compared with the traditional analytical method of linearization, the form of matrix analysis process is quite succinct and easy to understand, which can be used to the numerical solution of linear library application. For the various construction form of the rotation matrix, this method has better adaptability. The constructed matrix of collinear equation has important enlightenment significance for using computer vision method.

Key words collinear equation; projection equation; projection matrix; homogeneous coordinates

摄影测量学中，将同名的物点与像点满足的函数关系描述为共线方程。在不考虑影像畸变情况下，共线方程常表达为欧式坐标下的解析形式[1]:

$$\begin{cases} x = -f\dfrac{a_1\left(X-X_s\right)+b_1\left(Y-Y_s\right)+c_1\left(Z-Z_s\right)}{a_3\left(X-X_s\right)+b_3\left(Y-Y_s\right)+c_3\left(Z-Z_s\right)}, \\ y = -f\dfrac{a_2\left(X-X_s\right)+b_2\left(Y-Y_s\right)+c_2\left(Z-Z_s\right)}{a_3\left(X-X_s\right)+b_3\left(Y-Y_s\right)+c_3\left(Z-Z_s\right)}, \end{cases} \tag{1}$$

式中, $[x, y]^{\mathrm{T}}$ 为像点坐标, f 为主距, $[X, Y, Z]^{\mathrm{T}}$ 为物方点坐标, $[X_{\mathrm{S}}, Y_{\mathrm{S}}, Z_{\mathrm{S}}]^{\mathrm{T}}$ 为摄站坐标, $a_1 \sim c_3$ 为由外方位角元素构成的旋转矩阵元素。

共线方程贯穿于摄影测量学整个学科体系，数据处理时一般先对其进行线性化处理。由于解析形式下的共线方程式为非线性函数，且旋转矩阵元素与外方位角元素之间存在非常复杂的三角函数关系，所以在求导过程中即使利用多次变换技巧，像点坐标对外方位角元素求导后的解析结果也极其复杂。在计算机视觉领域，同名的物点与像点的函数关系称为投影方程，投影方程与共线方程仅个别参数定义有所差异[2]。投影方程一般描述为矩阵形式,

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通过矩阵分析方法获得相应问题结论，方程形式非常简洁。因此，可将共线方程也表达为矩阵形式，利用矩阵分析理论获得其线性化解。

1　齐次坐标描述的共线方程

在计算机视觉领域，习惯用齐次坐标下的矩阵形式表示投影方程[3–6]：

$$\lambda\tilde{\boldsymbol{x}}=\boldsymbol{M}\tilde{\boldsymbol{X}}\,, \tag{2}$$

或

$$\lambda\begin{bmatrix}x\\y\\1\end{bmatrix}=\begin{bmatrix}m_{11}&m_{12}&m_{13}&m_{14}\\m_{21}&m_{22}&m_{23}&m_{24}\\m_{31}&m_{32}&m_{33}&m_{34}\end{bmatrix}\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix}, \tag{3}$$

式中，$\boldsymbol{M}$ 为投影矩阵；λ 为尺度因子；$\tilde{\boldsymbol{x}}=\left[\boldsymbol{x}^{\mathrm{T}},\ 1\right]^{\mathrm{T}}=\left[x\quad y\quad 1\right]^{\mathrm{T}}$，$\tilde{\boldsymbol{x}}$ 为欧氏坐标下像点的物理坐标(mm)；$\tilde{\boldsymbol{X}}=\left[\boldsymbol{X}^{\mathrm{T}},1\right]^{\mathrm{T}}=\left[X\quad Y\quad Z\quad 1\right]^{\mathrm{T}}$，$\tilde{\boldsymbol{X}}$ 为欧氏坐标形式下同名物方点坐标。对比式(1)和(3)，显然，投影矩阵 $\boldsymbol{M}$ 包含主距 f、外方位元素线元素 X_{s} 及由外方位角元素构建的旋转矩阵 $\boldsymbol{R}$，因此，投影矩阵可构造为

$$\begin{aligned}\boldsymbol{M}&=\boldsymbol{K}\boldsymbol{R}^{\mathrm{T}}[\boldsymbol{I},-\boldsymbol{X}_{\mathrm{s}}]\\
&=\begin{bmatrix}-f&&\\&-f&\\&&1\end{bmatrix}\cdot\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}\cdot\begin{bmatrix}1&&&-X_{\mathrm{s}}\\&1&&-Y_{\mathrm{s}}\\&&1&-Z_{\mathrm{s}}\end{bmatrix}\\
&=\begin{bmatrix}-fa_1&-fb_1&-fc_1&fa_1X_{\mathrm{s}}+fb_1Y_{\mathrm{s}}+fc_1Z_{\mathrm{s}}\\-fa_2&-fb_2&-fc_2&fa_2X_{\mathrm{s}}+fb_2Y_{\mathrm{s}}+fc_2Z_{\mathrm{s}}\\a_3&b_3&c_3&-\left(a_3X_{\mathrm{s}}+b_3Y_{\mathrm{s}}+c_3Z_{\mathrm{s}}\right)\end{bmatrix}\\
&=\begin{bmatrix}m_{11}&m_{12}&m_{13}&m_{14}\\m_{21}&m_{22}&m_{23}&m_{24}\\m_{31}&m_{32}&m_{33}&m_{34}\end{bmatrix}。\end{aligned} \tag{4}$$

式中，$\boldsymbol{K}$ 为由主距构成的对角矩阵；$\boldsymbol{R}$ 为像空间坐标系到像空间辅助坐标系间的旋转矩阵；$\boldsymbol{X}_{\mathrm{s}}$ 为图像投影中心在像空间辅助坐标系中的欧氏坐标。

结合式(1)和(3)可以得到

$$\begin{cases}x=\dfrac{m_{11}X+m_{12}Y+m_{13}Z+m_{14}}{m_{31}X+m_{32}Y+m_{33}Z+m_{34}}=\dfrac{U}{\lambda},\\[2ex] y=\dfrac{m_{21}X+m_{22}Y+m_{23}Z+m_{24}}{m_{31}X+m_{32}Y+m_{33}Z+m_{34}}=\dfrac{V}{\lambda},\end{cases} \tag{5}$$

$$\begin{cases}U=m_{11}X+m_{12}Y+m_{13}Z+m_{14},\\V=m_{21}X+m_{22}Y+m_{23}Z+m_{24},\\\lambda=m_{31}X+m_{32}Y+m_{33}Z+m_{34}。\end{cases}$$

因此，式(1)~(3)和(5)在表达共线方程式时是一致的。

2　共线条件方程线性化的矩阵模型

以欧拉角描述的影像外方位元素(X_{s}，Y_{s}，Z_{s}，φ，ω，κ)为例，将式(1)线性化处理，得到像点坐标误差方程式：

$$\begin{cases}v_x=a_{11}\Delta X_{\mathrm{s}}+a_{12}\Delta Y_{\mathrm{s}}+a_{13}\Delta Z_{\mathrm{s}}+a_{14}\Delta\varphi+a_{15}\Delta\omega+\\\qquad a_{16}\Delta\kappa-a_{11}\Delta X-a_{12}\Delta Y-a_{13}\Delta Z-l_x,\\v_y=a_{21}\Delta X_{\mathrm{s}}+a_{22}\Delta Y_{\mathrm{s}}+a_{23}\Delta Z_{\mathrm{s}}+a_{24}\Delta\varphi+a_{25}\Delta\omega+\\\qquad a_{26}\Delta\kappa-a_{21}\Delta X-a_{22}\Delta Y-a_{23}\Delta Z-l_y。\end{cases} \tag{6}$$

式(6)可用矩阵表达为

$$\boldsymbol{V}_{\mathrm{p}}=\boldsymbol{A}\boldsymbol{X}_{\mathrm{EoP}}+\boldsymbol{B}\boldsymbol{X}_{\mathrm{Tie}}-\boldsymbol{L}_{\mathrm{p}}\,, \tag{7}$$

式中，$\boldsymbol{V}_{\mathrm{p}}=\left[v_x\quad v_y\right]^{\mathrm{T}}$，$\boldsymbol{L}_{\mathrm{p}}=\left[l_x\quad l_y\right]^{\mathrm{T}}$，

$$\boldsymbol{A}=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}&a_{16}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}&a_{26}\end{bmatrix},$$

$$\boldsymbol{B}=\begin{bmatrix}-a_{11}&-a_{12}&-a_{13}\\-a_{21}&-a_{22}&-a_{23}\end{bmatrix},$$

$\boldsymbol{X}_{\mathrm{EoP}}=\left[\Delta X_{\mathrm{s}}\quad\Delta Y_{\mathrm{s}}\quad\Delta Z_{\mathrm{s}}\quad\Delta\varphi\quad\Delta\omega\quad\Delta\kappa\right]^{\mathrm{T}}$，$\boldsymbol{X}_{\mathrm{Tie}}=[\Delta X\quad\Delta Y\quad\Delta Z]^{\mathrm{T}}$，$a_{11}$~$a_{26}$ 表示共线函数分别对各参数的偏导数。

这里，采用复合函数求导方法(链式法则)推导像点坐标对外方位元素的导数，即

$$\boldsymbol{A}=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}&a_{15}&a_{16}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}&a_{26}\end{bmatrix}$$

$$=\begin{bmatrix}\dfrac{\partial x}{\partial X_{\mathrm{s}}}&\dfrac{\partial x}{\partial Y_{\mathrm{s}}}&\dfrac{\partial x}{\partial Z_{\mathrm{s}}}&\dfrac{\partial x}{\partial\varphi}&\dfrac{\partial x}{\partial\omega}&\dfrac{\partial x}{\partial\kappa}\\[2ex]\dfrac{\partial y}{\partial X_{\mathrm{s}}}&\dfrac{\partial y}{\partial Y_{\mathrm{s}}}&\dfrac{\partial y}{\partial Z_{\mathrm{s}}}&\dfrac{\partial y}{\partial\varphi}&\dfrac{\partial y}{\partial\omega}&\dfrac{\partial y}{\partial\kappa}\end{bmatrix}$$

$$
=\begin{bmatrix}
\frac{\partial x}{\partial m_{11}} & \frac{\partial x}{\partial m_{12}} & \frac{\partial x}{\partial m_{13}} & \frac{\partial x}{\partial m_{14}} & \frac{\partial x}{\partial m_{21}} & \frac{\partial x}{\partial m_{22}} & \frac{\partial x}{\partial m_{23}} & \frac{\partial x}{\partial m_{24}} & \frac{\partial x}{\partial m_{31}} & \frac{\partial x}{\partial m_{32}} & \frac{\partial x}{\partial m_{33}} & \frac{\partial x}{\partial m_{34}} \\
\frac{\partial y}{\partial m_{11}} & \frac{\partial y}{\partial m_{12}} & \frac{\partial y}{\partial m_{13}} & \frac{\partial y}{\partial m_{14}} & \frac{\partial y}{\partial m_{21}} & \frac{\partial y}{\partial m_{22}} & \frac{\partial y}{\partial m_{23}} & \frac{\partial y}{\partial m_{24}} & \frac{\partial y}{\partial m_{31}} & \frac{\partial y}{\partial m_{32}} & \frac{\partial y}{\partial m_{33}} & \frac{\partial y}{\partial m_{34}}
\end{bmatrix}_{2\times 12}
$$

$$
\begin{bmatrix}
\frac{\partial m_{11}}{\partial X_s} & \frac{\partial m_{12}}{\partial X_s} & \frac{\partial m_{13}}{\partial X_s} & \frac{\partial m_{14}}{\partial X_s} & \frac{\partial m_{21}}{\partial X_s} & \frac{\partial m_{22}}{\partial X_s} & \frac{\partial m_{23}}{\partial X_s} & \frac{\partial m_{24}}{\partial X_s} & \frac{\partial m_{31}}{\partial X_s} & \frac{\partial m_{32}}{\partial X_s} & \frac{\partial m_{33}}{\partial X_s} & \frac{\partial m_{34}}{\partial X_s} \\
\frac{\partial m_{11}}{\partial Y_s} & \frac{\partial m_{12}}{\partial Y_s} & \frac{\partial m_{13}}{\partial Y_s} & \frac{\partial m_{14}}{\partial Y_s} & \frac{\partial m_{21}}{\partial Y_s} & \frac{\partial m_{22}}{\partial Y_s} & \frac{\partial m_{23}}{\partial Y_s} & \frac{\partial m_{24}}{\partial Y_s} & \frac{\partial m_{31}}{\partial Y_s} & \frac{\partial m_{32}}{\partial Y_s} & \frac{\partial m_{33}}{\partial Y_s} & \frac{\partial m_{34}}{\partial Y_s} \\
\frac{\partial m_{11}}{\partial Z_s} & \frac{\partial m_{12}}{\partial Z_s} & \frac{\partial m_{13}}{\partial Z_s} & \frac{\partial m_{14}}{\partial Z_s} & \frac{\partial m_{21}}{\partial Z_s} & \frac{\partial m_{22}}{\partial Z_s} & \frac{\partial m_{23}}{\partial Z_s} & \frac{\partial m_{24}}{\partial Z_s} & \frac{\partial m_{31}}{\partial Z_s} & \frac{\partial m_{32}}{\partial Z_s} & \frac{\partial m_{33}}{\partial Z_s} & \frac{\partial m_{34}}{\partial Z_s} \\
\frac{\partial m_{11}}{\partial \varphi} & \frac{\partial m_{12}}{\partial \varphi} & \frac{\partial m_{13}}{\partial \varphi} & \frac{\partial m_{14}}{\partial \varphi} & \frac{\partial m_{21}}{\partial \varphi} & \frac{\partial m_{22}}{\partial \varphi} & \frac{\partial m_{23}}{\partial \varphi} & \frac{\partial m_{24}}{\partial \varphi} & \frac{\partial m_{31}}{\partial \varphi} & \frac{\partial m_{32}}{\partial \varphi} & \frac{\partial m_{33}}{\partial \varphi} & \frac{\partial m_{34}}{\partial \varphi} \\
\frac{\partial m_{11}}{\partial \omega} & \frac{\partial m_{12}}{\partial \omega} & \frac{\partial m_{13}}{\partial \omega} & \frac{\partial m_{14}}{\partial \omega} & \frac{\partial m_{21}}{\partial \omega} & \frac{\partial m_{22}}{\partial \omega} & \frac{\partial m_{23}}{\partial \omega} & \frac{\partial m_{24}}{\partial \omega} & \frac{\partial m_{31}}{\partial \omega} & \frac{\partial m_{32}}{\partial \omega} & \frac{\partial m_{33}}{\partial \omega} & \frac{\partial m_{34}}{\partial \omega} \\
\frac{\partial m_{11}}{\partial \kappa} & \frac{\partial m_{12}}{\partial \kappa} & \frac{\partial m_{13}}{\partial \kappa} & \frac{\partial m_{14}}{\partial \kappa} & \frac{\partial m_{21}}{\partial \kappa} & \frac{\partial m_{22}}{\partial \kappa} & \frac{\partial m_{23}}{\partial \kappa} & \frac{\partial m_{24}}{\partial \kappa} & \frac{\partial m_{31}}{\partial \kappa} & \frac{\partial m_{32}}{\partial \kappa} & \frac{\partial m_{33}}{\partial \kappa} & \frac{\partial m_{34}}{\partial \kappa}
\end{bmatrix}_{6\times 12}^{\mathrm{T}} = \boldsymbol{J}_M \cdot \boldsymbol{J}_e \text{。} \tag{8}
$$

事实上，在数值解算中，只要弄清楚 $\boldsymbol{J}_{\mathrm{M}}$ 和 $\boldsymbol{J}_{\mathrm{e}}$，就可以得到系数矩阵 $\boldsymbol{A}$，这也是矩阵方法的另一个优势。

由式(5)易得

$$
\boldsymbol{J}_M = \begin{bmatrix}
\frac{\partial x}{\partial m_{11}} & \frac{\partial x}{\partial m_{12}} & \frac{\partial x}{\partial m_{13}} & \frac{\partial x}{\partial m_{14}} & \frac{\partial x}{\partial m_{21}} & \frac{\partial x}{\partial m_{22}} & \frac{\partial x}{\partial m_{23}} & \frac{\partial x}{\partial m_{24}} & \frac{\partial x}{\partial m_{31}} & \frac{\partial x}{\partial m_{32}} & \frac{\partial x}{\partial m_{33}} & \frac{\partial x}{\partial m_{34}} \\
\frac{\partial y}{\partial m_{11}} & \frac{\partial y}{\partial m_{12}} & \frac{\partial y}{\partial m_{13}} & \frac{\partial y}{\partial m_{14}} & \frac{\partial y}{\partial m_{21}} & \frac{\partial y}{\partial m_{22}} & \frac{\partial y}{\partial m_{23}} & \frac{\partial y}{\partial m_{24}} & \frac{\partial y}{\partial m_{31}} & \frac{\partial y}{\partial m_{32}} & \frac{\partial y}{\partial m_{33}} & \frac{\partial y}{\partial m_{34}}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\frac{X}{\lambda} & \frac{Y}{\lambda} & \frac{Z}{\lambda} & \frac{1}{\lambda} & 0 & 0 & 0 & 0 & -\frac{xX}{\lambda} & -\frac{xY}{\lambda} & -\frac{xZ}{\lambda} & -\frac{x}{\lambda} \\
0 & 0 & 0 & 0 & \frac{X}{\lambda} & \frac{Y}{\lambda} & \frac{Z}{\lambda} & \frac{1}{\lambda} & -\frac{yX}{\lambda} & -\frac{yY}{\lambda} & -\frac{yZ}{\lambda} & -\frac{y}{\lambda}
\end{bmatrix}
$$

$$
= \frac{1}{\lambda}\begin{bmatrix}
X & Y & Z & 1 & 0 & 0 & 0 & 0 & -xX & -xY & -xZ & -x \\
0 & 0 & 0 & 0 & X & Y & Z & 1 & -yX & -yY & -yZ & -y
\end{bmatrix}\text{。} \tag{9}
$$

为了求解 $\boldsymbol{J}_e$，对投影矩阵 $\boldsymbol{M}=\boldsymbol{K}\boldsymbol{R}^{\mathrm{T}}\left[\boldsymbol{I},-\boldsymbol{X}_{\mathrm{s}}\right]$ 两边对外方位元素求导可得

$$
\frac{\partial \boldsymbol{M}}{\partial(X_s, Y_s, Z_s, \varphi, \omega, \kappa)} = \boldsymbol{K}\frac{\partial \boldsymbol{R}^{\mathrm{T}}\left[\boldsymbol{I}, -\boldsymbol{X}_{\mathrm{s}}\right]}{\partial(X_s, Y_s, Z_s, \varphi, \omega, \kappa)}\text{。}
$$

$$
\frac{\partial \boldsymbol{M}}{\partial X_s} = \boldsymbol{K}\boldsymbol{R}^{\mathrm{T}}\frac{\partial\left[\boldsymbol{I}, -\boldsymbol{X}_{\mathrm{s}}\right]}{\partial X_s}
$$

$$
= \begin{bmatrix} -fa_1 & -fb_1 & -fc_1 \\ -fa_2 & -fb_2 & -fc_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 & 0 & fa_1 \\ 0 & 0 & 0 & fa_2 \\ 0 & 0 & 0 & -a_3 \end{bmatrix}\text{。} \tag{10}
$$

其他变量求导依此类推，

$$
\frac{\partial \boldsymbol{M}}{\partial Z_s} = \begin{bmatrix} 0 & 0 & 0 & fc_1 \\ 0 & 0 & 0 & fc_2 \\ 0 & 0 & 0 & -c_3 \end{bmatrix}，\ \frac{\partial \boldsymbol{M}}{\partial Y_s} = \begin{bmatrix} 0 & 0 & 0 & fb_1 \\ 0 & 0 & 0 & fb_2 \\ 0 & 0 & 0 & -b_3 \end{bmatrix}\text{。}
$$

对于 $\varphi-\omega-\kappa$ 转角系统，旋转矩阵

$$\boldsymbol{R}^{\mathrm{T}}=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}=\begin{bmatrix} \cos\varphi\cos\kappa-\sin\varphi\sin\omega\sin\kappa & \cos\omega\sin\kappa & \sin\varphi\cos\kappa+\cos\varphi\sin\omega\sin\kappa \\ -\cos\varphi\sin\kappa-\sin\varphi\sin\omega\cos\kappa & \cos\omega\cos\kappa & -\sin\varphi\sin\kappa+\cos\varphi\sin\omega\cos\kappa \\ -\sin\varphi\cos\omega & -\sin\omega & \cos\varphi\cos\omega \end{bmatrix},$$

则

$$\frac{\partial \boldsymbol{M}}{\partial\varphi}=\boldsymbol{K}\frac{\partial \boldsymbol{R}^{\mathrm{T}}}{\partial\varphi}\left[\boldsymbol{I},-\boldsymbol{X}_{\mathrm{S}}\right]=\boldsymbol{K}\boldsymbol{R}_{\varphi}^{\mathrm{T}}\left[\boldsymbol{I},-\boldsymbol{X}_{\mathrm{S}}\right]$$

$$=\begin{bmatrix} -f & & \\ & -f & \\ & & 1 \end{bmatrix}\cdot\begin{bmatrix} -c_1 & 0 & a_1 \\ -c_2 & 0 & a_2 \\ -c_3 & 0 & a_3 \end{bmatrix}\cdot\begin{bmatrix} 1 & & & -X_{\mathrm{s}} \\ & 1 & & -Y_{\mathrm{s}} \\ & & 1 & -Z_{\mathrm{s}} \end{bmatrix}=\begin{bmatrix} fc_1 & 0 & -fa_1 & -fc_1X_{\mathrm{s}}+fa_1Z_{\mathrm{s}} \\ fc_2 & 0 & -fa_2 & -fc_2X_{\mathrm{s}}-fa_2Z_{\mathrm{s}} \\ -c_3 & 0 & a_3 & c_3X_{\mathrm{s}}-a_3Z_{\mathrm{s}} \end{bmatrix}。\tag{11}$$

其他变量求导依此类推，可得

$$\frac{\partial \boldsymbol{M}}{\partial\omega}=\boldsymbol{K}\boldsymbol{R}_{\omega}^{\mathrm{T}}\left[\boldsymbol{I},-\boldsymbol{X}_{\mathrm{s}}\right]$$

$$=\begin{bmatrix} -fa_3\sin\kappa & -fb_3\sin\kappa & -fc_3\sin\kappa & f\sin\kappa(a_3X_{\mathrm{s}}+b_3Y_{\mathrm{s}}+c_3Z_{\mathrm{s}}) \\ -fa_3\cos\kappa & -fb_3\cos\kappa & -fc_3\cos\kappa & f\cos\kappa(a_3X_{\mathrm{s}}+b_3Y_{\mathrm{s}}+c_3Z_{\mathrm{s}}) \\ \sin\varphi\sin\omega & -\cos\omega & -\cos\varphi\sin\omega & -(\sin\varphi\sin\omega X_{\mathrm{s}}-\cos\omega Y_{\mathrm{s}}-\cos\varphi\sin\omega Z_{\mathrm{s}}) \end{bmatrix}。\tag{12}$$

$$\frac{\partial \boldsymbol{M}}{\partial\kappa}=\boldsymbol{K}\boldsymbol{R}_{\kappa}^{\mathrm{T}}\left[\boldsymbol{I},-\boldsymbol{X}_{s}\right]$$

$$=\begin{bmatrix} -f & & \\ & -f & \\ & & 1 \end{bmatrix}\cdot\begin{bmatrix} a_2 & b_2 & c_2 \\ -a_1 & -b_1 & -c_1 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & & & -X_{\mathrm{s}} \\ & 1 & & -Y_{\mathrm{s}} \\ & & 1 & -Z_{\mathrm{s}} \end{bmatrix}=\begin{bmatrix} -fa_2 & -fb_2 & -fc_2 & fa_2X_{\mathrm{s}}+fb_2Y_{\mathrm{s}}+fc_2Z_{\mathrm{s}} \\ fa_1 & fb_1 & fc_1 & -(fa_1X_{\mathrm{s}}+fb_1Y_{\mathrm{s}}+fc_1Z_{\mathrm{s}}) \\ 0 & 0 & 0 & 0 \end{bmatrix}。\tag{13}$$

通过以上式子可以得到 $\dfrac{\partial(m_{11},\ldots,m_{34})}{\partial(\varphi,\omega,\kappa)}$，从而得到 J_e。

由式(5)同样易得

$$\boldsymbol{J}_X=\begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \end{bmatrix}=\begin{bmatrix} \frac{m_{11}-xm_{31}}{\lambda} & \frac{m_{12}-xm_{32}}{\lambda} & \frac{m_{13}-xm_{33}}{\lambda} \\ \frac{m_{21}-ym_{31}}{\lambda} & \frac{m_{22}-ym_{32}}{\lambda} & \frac{m_{33}-ym_{33}}{\lambda} \end{bmatrix}$$

$$=\frac{1}{\lambda}\begin{bmatrix} m_{11}-xm_{31} & m_{12}-xm_{32} & m_{13}-xm_{33} \\ m_{21}-ym_{31} & m_{22}-ym_{32} & m_{33}-ym_{33} \end{bmatrix}$$

$$=\frac{1}{\lambda}\left\{\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{33} \end{bmatrix}-\begin{bmatrix} xm_{31} & xm_{32} & xm_{33} \\ ym_{31} & ym_{32} & ym_{33} \end{bmatrix}\right\}。\tag{14}$$

至此，基于矩阵分析方法的共线方程线性化系数推导完毕。需要指出的是，该方法对由不同姿态元素构造的旋转矩阵都有很好的适用性，如反对称矩阵法、四元数及轴角[7]等，差异之处仅在于投影矩阵对姿态元素变量的求导上，即式(11)~(13)。

由式(14)易知，对于物方点及外方位线元素的求导结果与解析方法一致，而对于外方位角元素，因篇幅有限，本文仅证明 $\dfrac{\partial x}{\partial\varphi}$，$\dfrac{\partial x}{\partial\omega}$，$\dfrac{\partial x}{\partial\kappa}$ 与解析法[1]的一致性。

3 矩阵分析法结论证明

1) $\dfrac{\partial x}{\partial \varphi}=\boldsymbol{J}_M\cdot\boldsymbol{J}_\varphi$

$$=\begin{bmatrix}\dfrac{\partial x}{\partial m_{11}} & \dfrac{\partial x}{\partial m_{12}} & \dfrac{\partial x}{\partial m_{13}} & \dfrac{\partial x}{\partial m_{14}} & \dfrac{\partial x}{\partial m_{21}} & \dfrac{\partial x}{\partial m_{22}} & \dfrac{\partial x}{\partial m_{23}} & \dfrac{\partial x}{\partial m_{24}} & \dfrac{\partial x}{\partial m_{31}} & \dfrac{\partial x}{\partial m_{32}} & \dfrac{\partial x}{\partial m_{33}} & \dfrac{\partial x}{\partial m_{34}}\end{bmatrix}\cdot$$

$$\begin{bmatrix}\dfrac{\partial m_{11}}{\partial \varphi} & \dfrac{\partial m_{12}}{\partial \varphi} & \dfrac{\partial m_{13}}{\partial \varphi} & \dfrac{\partial m_{14}}{\partial \varphi} & \dfrac{\partial m_{21}}{\partial \varphi} & \dfrac{\partial m_{22}}{\partial \varphi} & \dfrac{\partial m_{23}}{\partial \varphi} & \dfrac{\partial m_{24}}{\partial \varphi} & \dfrac{\partial m_{31}}{\partial \varphi} & \dfrac{\partial m_{32}}{\partial \varphi} & \dfrac{\partial m_{33}}{\partial \varphi} & \dfrac{\partial m_{34}}{\partial \varphi}\end{bmatrix}^{\mathrm{T}}$$

$$=\frac{1}{\lambda}\begin{bmatrix}X & Y & Z & 1 & 0 & 0 & 0 & 0 & -xX & -xY & -xZ & -x\end{bmatrix}\cdot$$

$$\begin{bmatrix}fc_1 & 0 & -fa_1 & -fc_1X_{\mathrm{s}}+fa_1Z_{\mathrm{s}} & fc_2 & 0 & -fa_2 & -fc_2X_{\mathrm{s}}+-fa_2Z_{\mathrm{s}} & -c_3 & 0 & a_3 & c_3X_{\mathrm{s}}-a_3Z_{\mathrm{s}}\end{bmatrix}^{\mathrm{T}}$$

$$=\frac{1}{\lambda}\left[f\left(X-X_{\mathrm{s}}\right)c_1+x\left(X-X_{\mathrm{s}}\right)\cdot c_3-f\left(Z-Z_{\mathrm{s}}\right)\cdot a_1-x\left(Z-Z_{\mathrm{s}}\right)a_3\right]。$$

考虑到 $\begin{bmatrix}X-X_{\mathrm{s}}\\ Y-Y_{\mathrm{s}}\\ Z-Z_{\mathrm{s}}\end{bmatrix}=\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}\begin{bmatrix}\bar{X}\\ \bar{Y}\\ \bar{Z}\end{bmatrix}$ 及 $\lambda=\bar{Z}$，则

$$\frac{\partial x}{\partial \varphi}=\frac{1}{\bar{Z}}\left[f\left(a_1\bar{X}+a_2\bar{Y}+a_3\bar{Z}\right)c_1+x\left(a_1\bar{X}+a_2\bar{Y}+a_3\bar{Z}\right)\cdot c_3-f\left(c_1\bar{X}+c_2\bar{Y}+c_3\bar{Z}\right)\cdot a_1-x\left(c_1\bar{X}+c_2\bar{Y}+c_3\bar{Z}\right)a_3\right]$$

$$=\frac{1}{\bar{Z}}\left[f\left(a_2\bar{Y}+a_3\bar{Z}\right)c_1-f\left(c_2\bar{Y}+c_3\bar{Z}\right)\cdot a_1+x\left(a_1\bar{X}+a_2\bar{Y}\right)\cdot c_3-x\left(c_1\bar{X}+c_2\bar{Y}\right)a_3\right]$$

$$=\frac{1}{\bar{Z}}\left[f\left(a_2c_1-c_2a_1\right)\bar{Y}+f\left(a_3c_1-c_3a_1\right)\bar{Z}+x\left(a_1c_3-c_1a_3\right)\bar{X}+x\left(a_2c_3-c_2a_3\right)\bar{Y}\right]$$

$$=\frac{1}{\bar{Z}}\left(fb_3\bar{Y}-fb_2\bar{Z}+xb_2\bar{X}-xb_1\bar{Y}\right)=-fb_3y-fb_2-\frac{x^2}{f}b_2+\frac{xy}{f}b_1。$$

2) $\dfrac{\partial x}{\partial \omega}=\boldsymbol{J}_M\cdot\boldsymbol{J}_\omega$

$$=\begin{bmatrix}\dfrac{\partial x}{\partial m_{11}} & \dfrac{\partial x}{\partial m_{12}} & \dfrac{\partial x}{\partial m_{13}} & \dfrac{\partial x}{\partial m_{14}} & \dfrac{\partial x}{\partial m_{21}} & \dfrac{\partial x}{\partial m_{22}} & \dfrac{\partial x}{\partial m_{23}} & \dfrac{\partial x}{\partial m_{24}} & \dfrac{\partial x}{\partial m_{31}} & \dfrac{\partial x}{\partial m_{32}} & \dfrac{\partial x}{\partial m_{33}} & \dfrac{\partial x}{\partial m_{34}}\end{bmatrix}\cdot$$

$$\begin{bmatrix}\dfrac{\partial m_{11}}{\partial \omega} & \dfrac{\partial m_{12}}{\partial \omega} & \dfrac{\partial m_{13}}{\partial \omega} & \dfrac{\partial m_{14}}{\partial \omega} & \dfrac{\partial m_{21}}{\partial \omega} & \dfrac{\partial m_{22}}{\partial \omega} & \dfrac{\partial m_{23}}{\partial \omega} & \dfrac{\partial m_{24}}{\partial \omega} & \dfrac{\partial m_{31}}{\partial \omega} & \dfrac{\partial m_{32}}{\partial \omega} & \dfrac{\partial m_{33}}{\partial \omega} & \dfrac{\partial m_{34}}{\partial \omega}\end{bmatrix}^{\mathrm{T}}$$

$$=\frac{1}{\lambda}\begin{bmatrix}X & Y & Z & 1 & 0 & 0 & 0 & 0 & -xX & -xY & -xZ & -x\end{bmatrix}\cdot$$

$$\begin{aligned}&[\ -fa_3\sin\kappa\quad -fb_3\sin\kappa\quad -fc_3\sin\kappa\quad f\sin\kappa(a_3X_{\mathrm{s}}+b_3Y_{\mathrm{s}}+c_3Z_{\mathrm{s}})\\ &\quad -fa_3\cos\kappa\quad -fb_3\cos\kappa\quad -fc_3\cos\kappa\quad f\cos\kappa(a_3X_{\mathrm{s}}+b_3Y_{\mathrm{s}}+c_3Z_{\mathrm{s}})\\ &\quad \sin\varphi\sin\omega\quad -\cos\omega\quad -\cos\varphi\sin\omega\quad -(\sin\varphi\sin\omega X_{\mathrm{s}}-\cos\omega Y_{\mathrm{s}}-\cos\varphi\sin\omega Z_{\mathrm{s}})]^{\mathrm{T}}\end{aligned}$$

$$\begin{aligned}=&\frac{1}{\lambda}\left\{f\sin\kappa[-a_3(X-X_{\mathrm{s}})-b_3(Y-Y_{\mathrm{s}})-c_3(Z-Z_{\mathrm{s}})]-x[\sin\varphi\sin\omega(X-X_{\mathrm{s}})-\right.\\ &\left.\cos\omega(Y-Y_{\mathrm{s}})-\cos\varphi\sin\omega(Z-Z_{\mathrm{s}})]\right\}\end{aligned}$$

$$\begin{aligned}=&\frac{1}{\bar{Z}}\left\{f\sin\kappa[-a_3(a_1\bar{X}+a_2\bar{Y}+a_3\bar{Z})-b_3(b_1\bar{X}+b_2\bar{Y}+b_3\bar{Z})-c_3(c_1\bar{X}+c_2\bar{Y}+c_3\bar{Z})]-\right.\\ &\left.x[\sin\varphi\sin\omega(a_1\bar{X}+a_2\bar{Y}+a_3\bar{Z})-\cos\omega(b_1\bar{X}+b_2\bar{Y}+b_3\bar{Z})-\cos\varphi\sin\omega(c_1\bar{X}+c_2\bar{Y}+c_3\bar{Z})]\right\}\end{aligned}$$

$$=\frac{1}{\bar{Z}}[-f\sin\kappa\bar{Z}+x(\bar{X}\sin\kappa+\bar{Y}\cos\kappa)]$$

$$=-f\sin\kappa-\frac{1}{f}(x^2\sin\kappa+xy\cos\kappa)\text{。}$$

3) $\frac{\partial x}{\partial\kappa}=\boldsymbol{J}_M\cdot\boldsymbol{J}_\kappa$

$$=\left[\frac{\partial x}{\partial m_{11}}\ \frac{\partial x}{\partial m_{12}}\ \frac{\partial x}{\partial m_{13}}\ \frac{\partial x}{\partial m_{14}}\ \frac{\partial x}{\partial m_{21}}\ \frac{\partial x}{\partial m_{22}}\ \frac{\partial x}{\partial m_{23}}\ \frac{\partial x}{\partial m_{24}}\ \frac{\partial x}{\partial m_{31}}\ \frac{\partial x}{\partial m_{32}}\ \frac{\partial x}{\partial m_{33}}\ \frac{\partial x}{\partial m_{34}}\right]\cdot$$

$$\left[\frac{\partial m_{11}}{\partial\kappa}\ \frac{\partial m_{12}}{\partial\kappa}\ \frac{\partial m_{13}}{\partial\kappa}\ \frac{\partial m_{14}}{\partial\kappa}\ \frac{\partial m_{21}}{\partial\kappa}\ \frac{\partial m_{22}}{\partial\kappa}\ \frac{\partial m_{23}}{\partial\kappa}\ \frac{\partial m_{24}}{\partial\kappa}\ \frac{\partial m_{31}}{\partial\kappa}\ \frac{\partial m_{32}}{\partial\kappa}\ \frac{\partial m_{33}}{\partial\kappa}\ \frac{\partial m_{34}}{\partial\kappa}\right]^{\mathrm{T}}$$

$$=\frac{1}{\lambda}\begin{bmatrix}X & Y & Z & 1 & 0 & 0 & 0 & 0 & -xX & -xY & -xZ & -x\end{bmatrix}\cdot$$

$$\begin{bmatrix}-fa_2 & -fb_2 & -fc_2 & fa_2X_s+fb_2Y_s+fc_2Z_s & fa_1 & fb_1 & fc_1 & -(fa_1X_s+fb_1Y_s+fc_1Z_s) & 0 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$$

$$=\frac{1}{\lambda}[-fa_2(X-X_s)-fb_2(Y-Y_s)-fc_2(Z-Z_s)]$$

$$=\frac{1}{\bar{Z}}\left[-f(a_2a_2+b_2b_2+c_2c_2)\bar{Y}\right]$$

$$=y\text{。}$$

求导结果与解析方法[1]的求导结果完全一致。

4　结论与贡献

共线方程线性化是摄影测量数据处理的前提，基于解析方法建立的共线方程线性化过程及结果形式非常复杂，不易掌握。利用矩阵分析方法能有效降低推导的复杂性，求导过程清晰明了，为摄影测量这一基本关系式处理提供全新的分析方法。本文主要结论和贡献如下。

1) 借鉴计算机视觉中“共线方程”的矩阵形式，构造出摄影测量中共线方程的矩阵表达形式，为利用矩阵分析方法奠定理论基础，也为计算机视觉与摄影测量学科之间建立重要联系。

2) 详细推导建立共线方程线性化的矩阵分析方法，并且对推导结果给予验证。

3) 矩阵分析方法建立的共线方程线性化过程对于其他形式构造的旋转矩阵都有较好的适用性，如四元数、反对称矩阵及轴角方法等，都可将共线方程线性化问题归结为旋转矩阵对角元素的求导，思路清晰，也利于数值解算。

参考文献

[1] 王之卓. 摄影测量原理. 北京: 测绘出版社, 1980

[2] 张祖勋. 数字摄影测量与计算机视觉. 武汉大学学报: 信息科学版, 2004, 29(12): 1035–1039

[3] 马颂德, 张正友. 计算机视觉——理论与算法. 北京: 科学出版社, 1998

[4] Hartley B R, Zisserman A. Multiple view geometry in computer vision. Cambridge: Cambridge University Press, 2003

[5] Szeliski R. Computer vision: algorithms and applications. Heidelberg: Springer-Verlag, 2011

[6] Forsyth D A, Ponce J. Computer vision: a modern approach. 2nd. New Jersey: Prentice Hall, 2012

[7] 徐振亮. 轴角描述的车载序列街景影像空中三角测量与三维重建方法研究[D]. 武汉: 武汉大学, 2014