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Noether Symmetries and Their Inverse Problems of Nonholonomic Systems with Fractional Derivatives

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Abstract Noether symmetries and their inverse problems of the nonholonomic systems with the fractional derivatives are studied. Based on the quasi-invariance of fractional Hamilton action under the infinitesimal transformations without the time and the general transcoordinates of time-reparametrization, the fractional Noether theorems are established for the nonholonomic constraint systems. Further, the fractional Noether inverse problems are firstly presented for the nonholonomic systems. An example is designed to illustrate the applications of the results.

Key words fractional derivative; nonholonomic system; Noether symmetry; Noether inverse problem

分数阶非完整系统的 Noether 对称性及其逆问题

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摘要 研究分数阶非完整系统的 Noether 对称性及其逆问题。基于分数阶非完整系统的 Hamilton 作用量关于广义坐标以及时间在无限小变换下的不变性, 提出系统的 Noether 定理, 并首次提出分数阶非完整动力学系统的逆问题。最后给出一个算例, 以说明结果的应用。

关键词 分数阶导数; 非完整系统; Noether 对称性; Noether 逆问题

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Fractional calculus is the emerging mathematical field dealing with the generalization of the derivatives and integrals to arbitrary real order. It was born in 1965 and from then on considered as the branches of mathematical and theoretical with no applications for many years. But, during the last two decades, it has been applied to many areas such as mathematics, economics, biology, engineering and physics[1–5]. Besides, it has played a significant role in quantum mechanics, long-range dissipation, electromagnetic theory, chaotic dynamics, and signal processing[6–11]. However, one can find its importance in the fractional of variations theory and optimal control. Riewe[12–13] studied a version of the Euler-Lagrange equations of conservative and nonconservative systems with fractional derivatives. Agrawal[14–15] obtained the Euler-Lagrange equations for fractional variational problems by using the fractional derivatives of Riemann-Liouville sense and Caputo sense. Then, El-Nabulsi[16–18], Ricardo et al.[19] and Teodor et al.[20] also made lots of contributions to the fractional variational problems.

The concepts of symmetry and conservation law are fundamental notions in physics and mathema-

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tics[21]. Symmetries are the invariance of the dynamical systems under the infinitesimal transformations, and hold the same object when applying the transformations. They are described mathematically by infinitesimal parameter group of transformations. The concept of symmetries of mechanical systems can be used to integrate the equations of motion and establish the invariance of the systems. They have played an important role in mathematics, physics, optimal control, engineering[22–30]. Conservation law of systems can be used to reduce to the dimension of the equations of motion and simplifying the resolution of the problems[31–32]. In the last few years, Fu et al.[33–35] , Zhang[36], and Li [37] made many important results symmetries and conserved quantities of nonholonomic systems. Zhou et al.[38] studied the Noether symmetry theories of the fractional Hamiltonian systems. Frederico et al.[39], Zhang[40–41], and Agrawal[42] also present the problems of Noether symmetry of fractional systems.

We all know that the fractional nonholonomic constraints restrict the stations of fractional systems, and the fractional nonholonomic systems are more generalize dynamical systems, which have attracted much attention. Sun et al.[43] gave the fractional first-order and second-order extensions form of Lie group transformation, and the corresponding Lie symmetries of fractional nonholonomic systems were discussed. Zhang et al.[44] studied Noether symmetries of fractional mechanico-electrical systems. Recently, Fu et al.[45] presented Lie symmetries and their inverse problems of fractional nonholonomic systems. However, applying fractional calculus to fractional nonholonomic systems and obtaining Noether inverse problem of nonholonomic systems have not been studied.

In this paper, we study the Noether symmetries and their inverse problems of nonholonomic systems with the fractional derivatives. Firstly, we establish the fractional derivatives equations of nonholonomic systems. Then, the Noether theorems and the corresponding conserved quantities are given by using the infinitesimal transformations without time and the general transformations of time-reparametrization. Finally, we study fractional Noether inverse problems.

1 Definitions and Properties of Riemann-Liouville Fractional Derivatives

In this section, we briefly recall some basic definitions and properties of left and right Riemann-Liouville fractional derivatives[40–41].

Definition 1 Let f be a continuous and integrable function in the interval $[a, b]$. The left Riemann-Liouville fractional derivatives (LRLFD) ${}_aD_t^\alpha f(t)$ and the right Riemann-Liouville fractional derivatives (RRLFD) ${}_tD_b^\alpha f(t)$ are defined as

$$\begin{aligned}&{}_aD_t^\alpha f(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_a^t(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau,\end{aligned}\tag{1}$$

$$\begin{aligned}&{}_tD_b^\alpha f(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_t^b(\tau-t)^{n-\alpha-1}f(\tau)\mathrm{d}\tau,\end{aligned}\tag{2}$$

where α is the order of the derivatives such that $n-1\leqslant\alpha<n$, $n\in\mathbf{N}$, and Γ is the Euler gamma function. If α is an integer, these derivatives are defined in the usual sense, i.e.

$${}_aD_t^\alpha f(t)=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t),$$

$${}_tD_b^\alpha f(t)=\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha f(t).$$

Theorem 1 Let f and g be two continuous functions defined on the interval $[a, b]$. Then for all $t\in[a, b]$, the following properties hold:
for $m>0$,

$${}_aD^m\left[f(t)+g(t)\right]={}_aD_t^m f(t)+{}_aD_t^m g(t)\,;\tag{3}$$

for $m\geqslant n\geqslant 0$,

$${}_aD_t^m\left({}_aD_t^{-n}f(t)\right)={}_aD_t^{m-n}f(t)\,;\tag{4}$$

for $m>0$,

$$\int_a^b ({}_aD_t^m f(t))g(t)\mathrm{d}t = \int_a^b f(t)({}_tD_b^m g(t))\mathrm{d}t\,. \tag{5}$$

From Agrawal[14], the Euler-Lagrange equations of conservative systems with the fractional variational problems as

$$\begin{aligned}&\frac{\partial L}{\partial q} + {}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q} + {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q} = 0,\\ &t \in [a,b],\end{aligned} \tag{6}$$

where L is a Lagrangian. When $\alpha = \beta = 1$, we have ${}_aD_t^\alpha = \frac{\mathrm{d}}{\mathrm{d}t}$ and ${}_tD_b^\beta = -\frac{\mathrm{d}}{\mathrm{d}t}$ and the Eq. (6) is reduce to the standard Euler-Lagrange equations.

2 The Equations of Motion of Nonholonomic Systems with Fractional Derivatives

In this section, we introduce the equations of motion and the Hamilton action of fractional nonholonomic systems[38]. At first, we consider the constrained mechanical system which configuration are determined by n generalized coordinates $q_k (k \in N, k = 1, 2, ..., n)$ and the motions of system are subjected to the μ ideal bilateral nonholonomic constraints of Appell-Chetaev type

$$\begin{aligned}&f_\mu = (t, q, {}_aD_t^\alpha q, {}_tD_b^\beta q)\\ &(\mu = 1, \ \ldots, \ g),\end{aligned} \tag{7}$$

we suppose that these constraints are independent each other, therefore the restrictive conditions of virtual displacement which decide on these constrains as follows:

$$\sum_{k=1}^n \frac{\partial f_\mu}{\partial_a D_t^\alpha q_k}\delta q_k = 0, \ \ \sum_{k=1}^n \frac{\partial f_\mu}{\partial_t D_b^\beta q_k}\delta q_k = 0\,. \tag{8}$$

Hence, the equations of motion of nonholonomic systems with fractional derivatives are given by

$$\begin{aligned}&\frac{\partial L}{\partial q_k} + {}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k} + {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\\ &= -Q_k - \sum_{k=1}^n \lambda_\mu \frac{\partial f_\mu}{\partial_a D_t^\alpha q_k} - \sum_{k=1}^n \lambda_\mu \frac{\partial f_\mu}{\partial_t D_b^\beta q_k},\end{aligned} \tag{9}$$

where L is the Lagrange function of the given systems, the Lagrangian $L:[a,b]\times R^n \times R^n \to R$ is determined by n generalize coordinates q_k and assumed to be a C^2 function with respect to all its arguments. The parameter λ is the Lagrange constraint multiplier, and Q_k is the non-potential generalized force.

When we assume that the fractional system is nonsingular, before calculus the derivatives function (7) and (9), we can get function $\lambda = \lambda(t, q, {}_aD_t^\alpha q, {}_tD_b^\beta q)$, therefore the Eq. (9) can be written as

$$\begin{aligned}&\frac{\partial L}{\partial q_k} + {}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k} + {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\\ &= -Q_k - \Lambda_k,\end{aligned} \tag{10}$$

where Λ_k is the nonholonomic constraint forces which determined by parameter $t, q, {}_aD_t^\alpha q, {}_tD_b^\beta q$, that is

$$\begin{aligned}\Lambda_k &= \Lambda_k(t, q, {}_aD_t^\alpha q, {}_tD_b^\beta q)\\ &= \sum_{k=1}^n \lambda \frac{\partial f_\mu}{\partial_a D_t^\alpha q_k} + \sum_{s=1}^n \lambda \frac{\partial f_\mu}{\partial_t D_b^\beta q_k},\end{aligned} \tag{11}$$

We say that the extremum problem of the faction integral function (12) is the fractional Hamilton action of the nonholonmic systems

$$S = \int_a^b L(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k)\,\mathrm{d}t\,, \tag{12}$$

with the commutative relations,

$$\begin{cases}\delta\, {}_aD_t^\alpha q_k = {}_aD_t^\alpha \delta q_k,\\ \delta\, {}_tD_b^\beta q_k = {}_tD_b^\beta \delta q_k,\end{cases} \tag{13}$$

and the boundary conditions,

$$q_k(a) = q_k\big|_{t=a}, \ q_k(b) = q_k\big|_{t=b},$$

where δ is the isochronous variation operator. The quasi-invariance problem of function (12) is called variational problem of fractional nonhonholonomic systems. When $\alpha = \beta = 1$, the problem is reduced to the classical Hamilton action variational problem of nonholonomic systems.

3 Noether Theorem of Nonholonomic Systems with Fractional Derivatives

In this section, we give the definition and the necessary conditions of the quasi-invariance of Hamilton action (12) under the infinitesimal group of transformations. We adopt the infinitesimal transformations contain without the time variable and the general transformations of time-reparametrization. Then we obtain the factional Noether theorems without transformation of the time and the general with transformation of time-reparametrization respectively.

Definition 2 (invariance without transforming the time). For a fractional nonholonomic system, we call that the formula (12) is quasi-invariant under the one-group of infinitesimal transformations

$$\bar{q}_k(t) = q_k(t) + \varepsilon\xi_k(t, q_k) + o(\varepsilon) \quad (k = 1, 2, \dots, n), \tag{14}$$

if and only if,

$$\begin{aligned}&\int_{t_1}^{t_2} L(t, q_k(t), {}_aD_t^\alpha q_k(t), {}_tD_b^\beta q_k(t))\mathrm{d}t \\ &= \int_{t_1}^{t_2} L(t, \bar{q}_k(t), {}_aD_t^\alpha \bar{q}_k(t), {}_tD_b^\beta \bar{q}_k(t))\mathrm{d}t + \\ &\int_{t_1}^{t_2} ({}_aD_t^\alpha \Delta G - {}_tD_b^\beta \Delta G + (Q_k + \Lambda_k)\delta q_k)\mathrm{d}t,\end{aligned} \tag{15}$$

for any subinterval $[t_1, t_2] \subset [a, b]$, where $\delta q_k = \varepsilon\xi_k$, ξ_k are the fractional infinitesimal generation functions of the infinitesimal transformations, $\Delta G = \varepsilon G_N(t, q_k)$ are fractional gauge functions of nonholonomic system.

Theorem 2 (Necessary condition of quasi-invariant). For a fractional nonholonomic system, if the function (12) is quasi-invariant under ε parameter infinitesimal group of transformations (14), then they must satisfy the following conditions,

$$\begin{aligned}&\frac{\partial L}{\partial q_k}\xi_k + \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}\, {}_aD_t^\alpha \xi_k + \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\, {}_tD_b^\beta \xi_k \\ &= {}_tD_b^\beta G_N - {}_aD_t^\alpha G_N - (Q_k + \Lambda_k)\xi_k.\end{aligned} \tag{16}$$

Proof By hypothesis, we know that the conditions (15) are true of the arbitrary subinterval $[t_1, t_2] \subset [a, b]$, taking the derivative of the condition (15) with respect to ε, substituting $\varepsilon = 0$. From the definitions and properties of the fractional derivatives, we get

$$\begin{aligned}0 = {}&\frac{\partial L}{\partial q_k}\xi_k + (Q_k + \Lambda_k)\xi_k + \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}\cdot \\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\theta)^{n-\alpha-1} q(\theta)\mathrm{d}\theta + \frac{\varepsilon}{\Gamma(n-\alpha)}\right. \\ &\left.\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\theta)^{n-\alpha-1}\xi_k(\theta, q)\mathrm{d}\theta\right]_{\varepsilon=0} + \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\cdot \\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{1}{\Gamma(n-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (\theta-t)^{n-\beta-1} q(\theta)\mathrm{d}\theta + \right. \\ &\left.\frac{\varepsilon}{\Gamma(n-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (\theta-t)^{n-\beta-1}\xi_k(\theta, q)\mathrm{d}\theta\right]_{\varepsilon=0} + \\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\frac{\varepsilon}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-\theta)^{n-\alpha-1} G_N(\theta, q)\mathrm{d}\theta\right]_{\varepsilon=0} - \\ &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\begin{array}{l}\dfrac{\varepsilon}{\Gamma(n-\beta)}\left(-\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^n \\ \displaystyle\int_t^b (\theta-t)^{n-\beta-1} G_N(\theta, q)\mathrm{d}\theta\end{array}\right]_{\varepsilon=0}.\end{aligned} \tag{17}$$

Eq. (17) is equivalent to Eq. (16).

In order to obtain the fractional conserved quantity of nonholonomic systems, we introduce the following definition[33].

Definition 3 Given two functions f and g of class C^1 in the interval $[a, b]$, we define the following operator:

$$\mathscr{D}_t^\alpha(f, g) = f\, {}_aD_t^\alpha g - g\, {}_tD_b^\alpha f, \quad t \in [a, b], \tag{18}$$

when $\alpha = 1$, operator $\mathscr{D}_t^\alpha$ is reduced to

$$\begin{aligned}\mathscr{D}_t^1(f, g) &= f\, {}_aD_t^1 g - g\, {}_tD_b^1 f \\ &= f\dot{g} + \dot{f}g = \frac{\mathrm{d}}{\mathrm{d}t} fg.\end{aligned}$$

Definition 4 (Fractional conserved quantity). For a fractional nonholonomic system, the function

$I = I(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k)$ is a fractional conserved quantity if and only it can be written as

$$I = \sum_{i=1}^{r} I_i^1(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k) \cdot I_i^2(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k), \tag{19}$$

where $r \in N$, and the pair I^1_i and I^2_i (i=1, …, r) must satisfy one of the following conditions:

$$\mathscr{D}_t^\alpha(I_i^1(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k), I_i^2(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k)) = 0,$$

or

$$\mathscr{D}_t^\alpha(I_i^2(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k), I_i^1(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k)) = 0,$$

under the fractional Euler-Lagrange Eq. (6).

Theorem 3 (Noether theorem without transformation of time). For a fractional nonholonomic system, if the Hamilton action satisfies the Definition 2 and ξ_k satisfies the necessary conditions (16), then the system possesses the fractional conserved quantity as follows:

$$I(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k) = \left(\frac{\partial L}{\partial\, {}_aD_t^\alpha q_k} - \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\right)\xi_k + G_N. \tag{20}$$

Proof We consider the fractional derivatives Eq. (10) of the nonholonomic systems:

$$\frac{\partial L}{\partial q_k} = -{}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k} - {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q_k} - (Q_k + \Lambda_k), \tag{21}$$

Substituting Eq. (21) into the necessary conditions of quasi-invariance (16), we obtain

$$\begin{aligned}
&-{}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}\xi_k - {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q}\xi_k - \\
&(Q_k + \Lambda_k)\xi_k + \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}\, {}_aD_t^\alpha \xi_k + \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\, {}_tD_b^\beta \xi_k + \\
&{}_aD_t^\alpha G_N - {}_tD_b^\beta G_N + (Q_k + \Lambda_k)\xi_k \\
&= \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}\, {}_aD_t^\alpha \xi_k - {}_tD_b^\alpha \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}\xi_k - \\
&\left({}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q}\xi_k - \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\, {}_tD_b^\beta \xi_k\right) + \\
&1 \cdot {}_aD_t^\alpha G_N - G_N\, {}_tD_b^\alpha 1 + G_N\, {}_aD_t^\beta 1 - {}_tD_b^\beta G_N \cdot 1 \\
&= \mathscr{D}_t^\alpha\left(\frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}, \xi_k\right) - \mathscr{D}_t^\beta\left(\xi_k, \frac{\partial L}{\partial\, {}_tD_b^\beta q_k}\right) + \\
&\mathscr{D}_t^\alpha(1, G_N) + \mathscr{D}_t^\beta(G_N, 1) \\
&= 0.
\end{aligned} \tag{22}$$

Definition 5 (Invariance of Eq. (12)). For a fractional nonholonomic system, we say that Eq. (12) is quasi-invariant under a ε parameter infinitesimal group of transformations

$$\begin{aligned}
&\bar{t} = t + \varepsilon\xi(t, q_k) + o(\varepsilon), \\
&\bar{q}_k(t) = q_k(t) + \varepsilon\xi_k(t, q_k) + o(\varepsilon) \\
&(k = 1, 2, ..., n),
\end{aligned} \tag{23}$$

if and only if

$$\begin{aligned}
&\int_{t_1}^{t_2} L(t, q_k(t)\, {}_aD_t^\alpha q_k(t), {}_tD_b^\beta q_k(t))\mathrm{d}t \\
&= \int_{t_1}^{t_2} (L(\bar{t}, \bar{q}_k(\bar{t}), {}_aD_t^\alpha \bar{q}_k(\bar{t}), {}_tD_b^\beta \bar{q}_k(\bar{t}))\mathrm{d}\bar{t} + \\
&\int_{t_1}^{t_2} ({}_aD_t^\alpha \Delta\bar{G} - {}_tD_b^\beta \Delta\bar{G}) + (Q_k + \Lambda_k)\delta\bar{q}_k)\mathrm{d}\bar{t},
\end{aligned} \tag{24}$$

for any subinterval $[t_1, t_2] \subset [a, b]$, where ξ be a infinitesimal generation functions of the infinitesimal transformations, $\delta\bar{q}_k = \varepsilon(\xi_k - \alpha\, {}_aD_t^\alpha q_k\xi) = \varepsilon(\xi_k + \beta\, {}_tD_b^\beta q_k\xi))$.

Theorem 4 (Noether theorem). For a fractional nonholonomic system, if Eq. (12) satisfies Definition 5 under the one-parameter group of infinitesimal transformations (23) and conditions (24), then the system holds the fractional conserved quantity as:

$$\begin{aligned}
&I(t, q_k, {}_aD_t^\alpha q_k, {}_tD_b^\beta q_k) \\
&= L\xi + \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}(\xi_k - \alpha\, {}_aD_t^\alpha q_k\xi) - \\
&\frac{\partial L}{\partial\, {}_tD_b^\beta q_k}(\xi_k + \beta\, {}_tD_b^\beta q_k\xi) + G_N.
\end{aligned} \tag{25}$$

Proof Introducing a one-to-one Lipschitzian transformation with respect to the independent variable t,

$$t \in [a,b] \mapsto \sigma f(\tau) \in [\sigma_a, \sigma_b], \tag{26}$$

and it satisfies if $\tau = 0$, then $t_\sigma{}' = f(\tau) = 1$. Applying this transformation into Eq. (24), we have

$$\begin{aligned} &\bar{S}(t(\cdot), q(t(\cdot)) \\ &= \int_{\sigma_a}^{\sigma_b} L\left\{ \begin{array}{l} t(\sigma), q_k(t(\sigma)), {}_aD_t^\alpha q_k(t(\sigma)), \\ {}_tD_b^\beta q_k(t(\sigma)) \end{array} \right\} t'_\sigma \mathrm{d}\sigma - \\ &\int_{t_1}^{t_2} \left\{ \begin{array}{l} {}_aD_t^\alpha \Delta G - {}_tD_b^\beta \Delta G + \\ (Q_k + \Lambda_k)\delta \bar{q}_k(t(\sigma)) \end{array} \right\} t'_\sigma \mathrm{d}\sigma, \end{aligned} \tag{27}$$

where $t(\sigma_a) = a, t(\sigma_b) = b$. Under the definitions of fractional Riemann-Liouville, we get

$$\begin{aligned} &{}_{\sigma_a}D_{t(\sigma)}^\alpha q_k\left(t(\sigma)\right) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t(\sigma)}\right)^n \cdot \\ &\int_{\frac{a}{f(\lambda)}}^{\sigma f(\lambda)} \left(\sigma f(\lambda) - \theta\right)^{n-\alpha-1} q_k\left(\theta f^{-1}(\lambda)\right) \mathrm{d}\theta \\ &= \frac{(t'_\sigma)^{-\alpha}}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^n \int_{\frac{a}{(t'_\sigma)^2}}^{\sigma} \left(\sigma - s\right)^{n-\alpha-1} q_k(s)\mathrm{d}s \\ &= (t'_\sigma)^{-\alpha} {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma). \end{aligned} \tag{28}$$

We can also obtain the following equality:

$${}_{t(\sigma)}D_{\sigma_b}^\beta q_k(t(\sigma)) = (t'_\sigma)^{-\beta} {}_\sigma D_{\frac{b}{(t'_\sigma)^2}}^\beta q_k(\sigma). \tag{29}$$

When $\lambda = 0$, we have

$$\begin{aligned} &(t'_\sigma)^{-\alpha} {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma) \\ &= {}_aD_t^\alpha q_k, (t'_\sigma)^{-\beta} {}_\sigma D_{\frac{b}{(t'_\sigma)^2}}^\beta q_k(\sigma) \\ &= {}_tD_b^\beta q_k. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\bar{S}(t(\cdot), q(t(\cdot)) \\ &= \int_{\sigma_a}^{\sigma_b} L\left\{ \begin{array}{l} t(\sigma), q_k(t(\sigma)), (t'_\sigma)^{-\alpha} \\ {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma), (t'_\sigma)^{-\beta} {}_\sigma D_{\frac{b}{(t'_\sigma)^2}}^\beta q_k(\sigma) \end{array} \right\} t'_\sigma \mathrm{d}\sigma - \\ &\int_{t_1}^{t_2} \left\{ \begin{array}{l} {}_aD_t^\alpha (\varepsilon G_N(t(\sigma), q_k(t(\sigma)))) - \\ {}_tD_b^\beta (\varepsilon G_N(t(\sigma), q_k(t(\sigma)))) \end{array} \right\} t'_\sigma \mathrm{d}\sigma - \\ &\int_{t_1}^{t_2} \left\{ \begin{array}{l} (Q_k + \Lambda_k) \cdot \varepsilon(\xi_k - \alpha(t'_\sigma)^{-\alpha} \\ {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma)\xi) \end{array} \right\} t'_\sigma \mathrm{d}\sigma \\ &= \int_{\sigma_a}^{\sigma_b} \bar{L}\left\{ \begin{array}{l} t(\sigma), q_k(t(\sigma)), t'_\sigma, (t'_\sigma)^{-\alpha} {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma), \\ (t'_\sigma)^{-\beta} {}_\sigma D_{\frac{b}{(t'_\sigma)^2}}^\beta q_k(\sigma) \end{array} \right\} \mathrm{d}\sigma - \\ &\int_{t_1}^{t_2} \left\{ \begin{array}{l} {}_aD_t^\alpha \Delta\bar{G}\left(t(\sigma), q_k(t(\sigma)), t'_\sigma\right) - \\ {}_tD_b^\beta \Delta\bar{G}(t(\sigma), q_k(t(\sigma)), t'_\sigma) \end{array} \right\} \mathrm{d}\sigma - \\ &\int_{t_1}^{t_2} \left\{ \begin{array}{l} (Q_k + \Lambda_k) \cdot \varepsilon(\xi_k - \alpha(t'_\sigma)^{-\alpha} \\ {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma)\xi) \end{array} \right\} t'_\sigma \mathrm{d}\sigma \\ &= \int_a^b L(t, q_k(t)\, {}_aD_t^\alpha q_k(t), {}_tD_b^\beta q_k(t))\, \mathrm{d}t - \\ &\int_{t_1}^{t_2} ({}_aD_t^\alpha \Delta G - {}_tD_b^\beta \Delta G + (Q_k + \Lambda_k)\delta\bar{q}_k)\, \mathrm{d}t \\ &= S(q(\cdot)). \end{aligned} \tag{30}$$

We know that if the functional (12) satisfies the quasi-invariant condition (24) under the sense of Definition 5, then Eq. (27) satisfies the quasi-invariant condition (15) under the sense of Definition 2. Finally using Theorem 3, we obtain the following fractional conserved quantity:

$$\begin{aligned} &I(t(\sigma), q_k(t(\sigma)), t'_\sigma, (t'_\sigma)^{-\alpha} {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma), \\ &(t'_\sigma)^{-\beta} {}_\sigma D_{\frac{a}{(t'_\sigma)^2}}^\beta q_k(\sigma)) = \left(\frac{\partial\bar{L}}{\partial(t'_\sigma)^{-\alpha} {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma)} - \right. \\ &\left. \frac{\partial\bar{L}}{\partial(t'_\sigma)^{-\beta} {}_\sigma D_{\frac{b}{(t'_\sigma)^2}}^\beta q_k(\sigma)} \right)\xi_k + \frac{\partial}{\partial t'_\sigma}\bar{L}\xi + G_N \\ &= L\xi + \frac{\partial L}{\partial\, {}_aD_t^\alpha q_k}(\xi_k - \alpha\, {}_aD_t^\alpha q_k\xi) - \\ &\frac{\partial L}{\partial\, {}_tD_b^\beta q_k}(\xi_k + \beta\, {}_tD_b^\beta q_k\xi) + G_N, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &\frac{\partial}{\partial t'_\sigma}\bar{L}\xi = L + \frac{\partial\bar{L}}{\partial(t'_\sigma)^{-\alpha} {}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma)} \cdot \frac{\partial}{\partial t'_\sigma} \cdot \\ &\left[\frac{(t'_\sigma)^{-\alpha}}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^n \int_{\frac{a}{(t'_\sigma)^2}}^{\sigma} (\sigma - s)^{n-\alpha-1} q_k(s)\mathrm{d}s \right] t'_\sigma + \\ &\frac{\partial\bar{L}}{\partial(t'_\sigma)^{-\beta} {}_\sigma D_{\frac{b}{(t'_\sigma)^2}}^\beta q_k(\sigma)} \cdot \end{aligned}$$

$$\frac{\partial}{\partial t'_\sigma}\left[\frac{(t'_\sigma)^{-\beta}}{\Gamma(n-\beta)}\left(-\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^n\int_\sigma^{\frac{b}{(t'_\sigma)^2}}(s-\sigma)^{n-\beta-1}q_k(s)\mathrm{d}s\right]t'_\sigma$$

$$=L-\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha q_k}\,{}_aD_t^\alpha q_k-\beta\frac{\partial L}{\partial\,{}_tD_b^\beta q_k}\,{}_tD_b^\beta q_k, \tag{32}$$

and

$$\frac{\partial L}{\partial\,{}_aD_t^\alpha q_k}-\frac{\partial L}{\partial\,{}_tD_b^\beta q_k}$$
$$=\frac{\partial\overline{L}}{\partial(t'_\sigma)^{-\alpha}\,{}_{\frac{a}{(t'_\sigma)^2}}D_\sigma^\alpha q_k(\sigma)}-\frac{\partial\overline{L}}{\partial(t'_\sigma)^{-\beta}\,{}_\sigma D^\beta_{\frac{b}{(t'_\sigma)^2}}q_k(\sigma)}. \tag{33}$$

4 Noether Inverse Problems of Nonholonomic Systems with Fractional Derivatives

In this section, we study the inverse problems of dynamics for the nonholonomic systems with fractional derivatives. By using Noether theory the generators and the gauge functions of the infinitesimal transformations corresponding to the known conserved quantities are deduced simultaneously.

Firstly, we suppose that nonholonomic system is nonsingular and the fractional conserved quantity is

$$I=I(t,q_k,{}_aD_t^\alpha q_k,{}_tD_b^\beta q_k)=\text{const}. \tag{34}$$

Let the fractional differential operator ${}_tD_b^\alpha$ act on Eq. (28), we obtain

$${}_tD_b^\alpha t\frac{\partial I}{\partial t}+{}_tD_b^\alpha q_k\frac{\partial I}{\partial q_k}+{}_tD_b^\alpha({}_aD_t^\alpha q_k)\frac{\partial I}{\partial_aD_t^\alpha q_k}+$$
$${}_tD_b^{\alpha+\beta}q_k\frac{\partial I}{\partial_tD_b^\beta q_k}, \tag{35}$$

and using the same method, from the fractional differential operator ${}_aD_t^\beta$, we have

$${}_aD_t^\beta t\frac{\partial I}{\partial t}+{}_aD_t^\beta q_k\frac{\partial I}{\partial q_k}+{}_aD_t^{\beta+\alpha}q_k\frac{\partial I}{\partial_aD_t^\alpha q_k}+$$
$${}_aD_t^\beta({}_tD_b^\beta q_k)\frac{\partial I}{\partial_tD_b^\beta q_k}. \tag{36}$$

Then, multiplying $\xi_k-\alpha_aD_t^\alpha q_k\xi$ on both side of Eq. (8) and expanding of the result, we get

$$\left(\frac{\partial L}{\partial q_k}+{}_tD_b^\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha q_k}+{}_aD_t^\beta\frac{\partial L}{\partial\,{}_tD_b^\beta q_k}+Q_k+\Lambda_k\right)\cdot$$
$$(\xi_k-\alpha_aD_t^\alpha q_k\xi)$$
$$=\left(\frac{\partial^2L}{\partial t\partial\,{}_aD_t^\alpha q_k}\,{}_tD_b^\alpha t+\frac{\partial^2L}{\partial q_k\partial\,{}_aD_t^\alpha q_k}\,{}_tD_b^\alpha q_k+\right.$$
$$\frac{\partial^2L}{\partial\,{}_tD_b^\beta q_k\partial\,{}_aD_t^\alpha q_k}\,{}_tD_b^{\alpha+\beta}q_k+\frac{\partial^2L}{\partial t\partial\,{}_tD_b^\beta q_k}\,{}_aD_t^\beta t+$$
$$\frac{\partial^2L}{\partial q_k\partial\,{}_tD_b^\beta q_k}\,{}_aD_t^\beta q_k+\frac{\partial^2L}{\partial\,{}_aD_t^\alpha q_k\partial\,{}_tD_b^\beta q_k}\,{}_aD_t^{\alpha+\beta}q_k)+$$
$$\frac{\partial^2L}{\partial({}_aD_t^\alpha q_k)^2}\,{}_tD_b^\alpha({}_aD_t^\alpha q_k)+\frac{\partial^2L}{\partial({}_tD_b^\beta q_k)^2}\cdot$$
$$\left.{}_aD_t^\beta({}_tD_b^\beta q_k+\frac{\partial L}{\partial q_k}+Q_k+\Lambda_k\right)(\xi_k-\alpha_aD_t^\alpha q_k\xi). \tag{37}$$

Using the similar multiplier $\xi_k+\beta_tD_b^\beta q_k\xi$, we can also expand the following formula:

$$\left(\frac{\partial L}{\partial q_k}+{}_tD_b^\alpha\frac{\partial L}{\partial\,{}_aD_t^\alpha q_k}+{}_aD_t^\beta\frac{\partial L}{\partial\,{}_tD_b^\beta q_k}+Q_k+\Lambda_k\right)\cdot$$
$$(\xi_k+\beta_tD_b^\beta q_k\xi), \tag{38}$$

Further, we use Eq. (37) minus (35), separate out the items of containing ${}_tD_b^\alpha({}_aD_t^\alpha q_k)$, and make its coefficient be equal to zero, we get

$$\frac{\partial^2L}{\partial({}_aD_t^\alpha q_k)^2}(\xi_k-\alpha_aD_t^\alpha q_k\xi)-\frac{\partial I}{\partial_aD_t^\alpha q_k}=0. \tag{39}$$

Using the same method, Eq. (38) minus (36), separating out the items of containing ${}_aD_t^\beta({}_tD_b^\beta q_k)$, and making its coefficient be equal to zero, we get

$$\frac{\partial^2L}{\partial({}_tD_b^\beta q_k)^2}(\xi_k+\beta_tD_b^\beta q_k\xi)-\frac{\partial I}{\partial_tD_b^\beta q_k}=0. \tag{40}$$

By hypothesis, we know the nonsingular of the given fractional nonholonomic system, from Eq. (39) and (40) , we obtain

$$\begin{cases}\overline{\xi}_1=\left(\dfrac{\partial^2L}{\partial({}_aD_t^\alpha q_k)^2}\right)^{-1}\dfrac{\partial I}{\partial_aD_t^\alpha q_k},\\ \overline{\xi}_2=\left(\dfrac{\partial^2L}{\partial({}_tD_b^\beta q_k)^2}\right)^{-1}\dfrac{\partial I}{\partial\,{}_tD_b^\beta q_k},\end{cases} \tag{41}$$

where

$$\overline{\xi}_1 = \xi_k - \alpha_a D_t^\alpha q_k \xi,\ \overline{\xi} = \xi_k + \beta_t D_b^\beta q_k \xi \, .$$

Finally, in order to obtain the infinitesimal generation function ξ and the gauge functions, let the function (34) be equal to the conserve quantity (Eq. (25)) determined by Theorem 4, we have

$$L\xi + \frac{\partial L}{\partial_a D_t^\alpha q_k}(\xi_k - \alpha_a D_t^\alpha q_k \xi) - \frac{\partial L}{\partial_t D_b^\beta q_k}(\xi_k + \beta_t D_b^\beta q_k \xi) + G_N = I. \tag{42}$$

Eqs. (41) and (42) reduce to the generation functions of infinitesimal transformations.

5 Example

We consider the kinetic energy and potential energy of the system respectively as follows:

$$T = \frac{1}{2}(({}_a D_t^\alpha q_1)^2 + ({}_a D_t^\alpha q_2)^2),\quad V = 0\,, \tag{43}$$

the nonholonomic constraint as:

$$f = {}_a D_t^\alpha q_1 + bt_a D_t^\alpha q_2 - bq_2 + t = 0\,. \tag{44}$$

Now we study its Noether symmetry and its inverse problems.

1) The Lagrangian of the nonhonolomic system is as follows:

$$L = T - V = \frac{1}{2}(({}_a D_t^\alpha q_1)^2 + ({}_a D_t^\alpha q_2)^2)\,, \tag{45}$$

the fractional Hamilton action can be written as

$$S = \int_a^b 1/2(({}_a D_t^\alpha q_1)^2 + ({}_a D_t^\alpha q_2)^2)\,\mathrm{d}t\,, \tag{46}$$

which is quasi-invariant under Definition 5. For the problem (46), we can conclude the following solutions from the condition (24):

$$\xi^1 = 0,\quad \xi_1^{\ 1} = -bt,\quad \xi_2^{\ 1} = 1,\quad G = -bq_1\,; \tag{47}$$

$$\xi^2 = 1,\quad \xi_1^{\ 2} = {}_a D_t^\alpha q_1,\quad \xi_2^{\ 2} = {}_a D_t^\alpha q_2,$$
$$G = \frac{1}{2}(({}_a D_t^\alpha q_1)^2 + ({}_a D_t^\alpha q_2)^2)\,. \tag{48}$$

Eqs. (47) and (48) corresponding to Noether symmetries of the fractional Hamilton action (46). For the fractional Noether Theorem 4, the fractional conserved quantities as follows (25),

$$I^1 = -bt\,{}_a D_t^\alpha q_1 + {}_a D_t^\alpha q_2 + bq_1\,, \tag{49}$$

$$I^2 = 0\,. \tag{50}$$

2) Noether inverse problems.

We suppose that

$$I = -bt\,{}_a D_t^\alpha q_1 + {}_a D_t^\alpha q_2 + bq_1\,, \tag{51}$$

is the fractional conserve quantity of the nonholonomic system, and the fractional Lagrangian is Eq. (45). Then by using Eqs. (41) and (42), the generators and the gauge functions of the infinitesimal transformations corresponding to the known conserved quantities are deduced simultaneously, we get

$$\begin{cases} \xi_1 - \alpha\,{}_a D_t^\alpha q_1 \xi = -bt, \\ \xi_2 + \beta\,{}_a D_t^\alpha q_2 \xi = 1, \\ L\xi - bt\,{}_a D_t^\alpha q_1 + {}_a D_t^\alpha q_2 + G_N \\ \quad = -bt\,{}_a D_t^\alpha q_1 + {}_a D_t^\alpha q_2 + bq_1, \end{cases} \tag{52}$$

the solution can be written as

$$\begin{cases} \xi = \frac{1}{L}(bq_1 - G), \\ \xi_1 = -bt + \alpha\,{}_a D_t^\alpha q_1 \xi, \\ \xi_2 = 1 - \beta\,{}_a D_t^\alpha q_2 \xi. \end{cases} \tag{53}$$

When G is given by

$$G = bq_1\,, \tag{54}$$

we get

$$\xi = 0,\quad \xi_1 = -bt,\quad \xi_2 = 1\,. \tag{55}$$

When G is given by,

$$G = bq_1 - L\,, \tag{56}$$

we get

$$\begin{cases} \xi = 1, \\ \xi_1 = -bt + \alpha\,{}_a D_t^\alpha q_1, \\ \xi_2 = 1 - \beta\,{}_a D_t^\alpha q_2. \end{cases} \tag{57}$$

6 Conclusion

In this paper, we use the Riemann-Liouville fractional derivatives to obtain the fractional Noether theorem and the fractional Noether inverse theorem of nonholonomic systems under the infinitesimal transformations. We find that the dynamic symmetry inverse problems of multiple values are the inherent characteristics, and how to choose the appropriate equations in practice need further research.

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